Towards Dynamical Qubit Controlling with Time-Dependent External Fields

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For a flux qubit described by a two-level system of equations we propose a special time dependent external control field. We show that for a qubit placed in this field there exists a critical value of tunnel frequency. When the tunnel frequency is close to its critical value, the probability value of a definite direction of the current circulating in a Josephson-junction circuit may be kept above 1/2 during a desirable time interval. We also show that such a behavior is not much affected by a sufficiently small dissipation.

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In the past few years superconducting circuits based on Josephson tunnel junction have attracted much attention both from theoretical and experimental viewpoints as possible candidates for the implementation of quantum computation (see, e.g. Refs. [1, 2, 3, 4]). Usually they represent a small Josephson-junction circuit, called a Cooper-pair box, which consists of a small superconducting electrode connected to a reservoir via a Josephson junction [2]. For a flux qubit the circuit with a very small inductance containing three Josephson junctions is described (in appropriate units) by the following two-level Hamiltonian [3, 4]:

$$H_q = -\Delta\sigma_x - \varepsilon(t)\sigma_z\,. \tag{1}$$

Here Δ is the tunnel frequency and $\varepsilon(t)$ is a time-dependent field (bias), which is controlled by an externally applied flux. Solving the Schrödinger equation with Hamiltonian (1), $i\dot{\Psi}(t) = H_q\Psi$, $\Psi = (\psi_1, \psi_2)^T$ (superscript "T" means the transposition and the dot over a symbol means the derivative with respect to time) we obtain the probability of a definite direction of the current circulating in the ring, i.e. $P^{\downarrow} = |\psi_1|^2$ is the probability of the clockwise direction of the current and $P^{\uparrow} = |\psi_2|^2$ is the probability of the opposite current direction.

One of the most important problem in quantum computation is connected with the possibility of controlling the state of an array of qubits. Typically the simplest two-qubit operations are generated by interplay of the coupling between qubits and local fields. Much theoretical attention has been recently paid to studying the controllable coupling between qubits of different types (see Ref. [5] and references therein). Recently it has been shown [6] that in the simplest and the most important from engineering viewpoint case of an "always on" and fixed coupling, a two-qubit Hamiltonian may be decoupled and the control problem is, in particular, reduced to finding the evolution of a one qubit placed in a time-dependent external control field. This observation shows an additional importance of controlling a one qubit state. This is the subject we devote the present note.

Usually, the probabilities, as functions of time, show an oscillating behavior (cf. famous Rabi oscillations, see e.g. Ref. [7]). But for some specific external fields this character may be changed drastically [8] thus showing a possibility to control the qubit state. Up to now such a possibility is known to be mostly related to oscillating external fields. Recently, using the method of supersymmetric quantum mechanics, new analytically solvable external fields, were indicated [9, 10], among which one can find smooth non-periodical functions. An advantage of analytic solutions is the possibility of a careful analysis of their properties which may reveal unexpected peculiarities [11]. In this note we apply these results to show the possibility of controlling the qubit state with an external field of a special configuration. We show that in this case there exists a critical value of the tunnel frequency Δ. While tunnel frequency approaches its critical value, probability $P^{\downarrow}(t)$ oscillates around a value exceeding 1/2 with a decreasing amplitude and after the critical value is reached it becomes a function monotonously increasing up to limiting value equal 3/4. Then using the property that this special excitation regime is, in fact, a limiting case of a more general oscillating external field we demonstrate that one can control a definite direction of the current in the ring (i.e. the qubit state) during a desirable time interval. Finally we also show that such a behavior is not much affected by the presence of a reasonably small dissipation featuring open quantum systems.

Consider first the case when the external control field $\varepsilon = \varepsilon_1(t)$ changes in the following way:

$$\varepsilon_1(t) = -\varepsilon_0 + \frac{4\varepsilon_0}{1 + 4\varepsilon_0^2 t^2}\,. \tag{2}$$

Parameter ε_0 gives us the possibility to choose a suitable time scale since after re-scaling $\tau = 2\varepsilon_0 t$, and redefining parameter Δ, $\Delta = 2\varepsilon_0\delta$, we obtain the Schrödinger equation with the Hamiltonian

$$H = -\delta\sigma_x - \epsilon(\tau)\sigma_z\,, \qquad \epsilon(\tau) = \epsilon_1(\tau) = -\frac{1}{2} + \frac{2}{1+\tau^2}\,, \tag{3}$$

for which exact analytic solutions are known [9]. Therefore imposing the initial condition $P^{\downarrow}(0) = 0$ we can write down an explicit expression for the probability

$P^{\downarrow}(\tau)$:

$$P_1^{\downarrow}(\tau) = \frac{(\theta^2-1)(\theta^2+4)}{2\theta^4}\frac{\tau^2}{1+\tau^2} + \frac{(\theta^2-1)(\theta^2-4)}{2\theta^6(1+\tau^2)} \times \left[\theta^2 - 4 - (\theta^2 - 4 + \theta^2\tau^2)\cos\theta\tau + 4\theta\tau\sin\theta\tau\right] \tag{4}$$

where we have introduced $\theta = \sqrt{1+4\delta^2}$. From here it is clearly seen that $P_1^{\downarrow}(\tau)$ is an oscillating function provided $\theta \neq 2$ ($\delta \neq \sqrt{3}/2$). For $\theta = 2$ ($\delta = \sqrt{3}/2$) Eq. (4) yields

$$P_1^{\downarrow}(\tau) = \frac{3}{4}\frac{\tau^2}{1+\tau^2}\,, \tag{5}$$

which is a function monotonously increasing from zero at the initial time till the value 3/4 for $\tau \gg 1$ (solid (black) line in Fig. 1a). We note a decrease of the oscillation amplitude when δ approaches its critical value equal $\sqrt{3}/2$. This is why for δ close enough to the value $\sqrt{3}/2$ the minimal value of the probability $P_1^{\downarrow}(\tau)$ for $\tau > 2$ exceeds 1/2 (see dashed (blue) and dotted (purple) lines in Fig. 1a).

The existence of a critical value of the tunnel frequency ($\delta = \sqrt{3}/2$ in this particular case; since δ differs from Δ only by a scaling factor $2\varepsilon_0$ we will call δ tunnel frequency as well) is reflected also by the time-averaged probability which exhibits a maximum (solid (black) line in Fig. 2a). Its analytic expression

$$\overline{P_1^{\downarrow}} = 2\delta^2\frac{5+4\delta^2}{(1+4\delta^2)^2} \tag{6}$$

allows us to get the exact position of the maximum which is $25/32 \approx 0.78$ at $\delta = \sqrt{5/12}$.

The result we have just obtained suggests us to consider a more general case [9, 10] where function $\epsilon(\tau)$ being periodical, depends on three parameters, one of which we fix by re-scaling both the time and another parameter (frequency ω appearing in Eq. (7)) in a way as it has been done above, thus obtaining the field

$$\epsilon_2(\tau) = -1/2 - \frac{2\omega^2}{b\cos(2\omega\tau+\varphi) - 1/2}\,. \tag{7}$$

Here $b^2 = 1/4 - ø^2 > 0$ and φ is the remaining parameter which also can be eliminated by shifting the time origin so that we put $\varphi = 0$, thus reducing external fields to a one parameter ($ø$) family. It is important to note that the previous result (3) may be obtained from here at $\varphi = 0$ in the limit $ø \to 0$.

According to Ref. [9] the analytic expression for $P_2^{\downarrow}(\tau)$ reads

$$\begin{aligned} P_2^{\downarrow}(\tau) &= \frac{4\delta^2}{\theta^2}\sin^2\left(\tfrac{1}{2}\theta\tau\right) \\ &- \frac{4\delta^2 b\left[Q - ø(b+b^2-\delta^2)\theta\sin(2ø\tau)\sin(\theta\tau)\right]}{\theta^2(b^2+\delta^2)^2(2b\cos(2ø\tau)-1)}\,, \end{aligned}$$

where

$$\begin{aligned} Q &= b(1+2b)\theta^2\cos^2\left(\tfrac{1}{2}\theta\tau\right)\sin^2(ø\tau) \\ &+ 4ø^2(b-2\delta^2)\cos^2(ø\tau)\sin^2\left(\tfrac{1}{2}\theta\tau\right). \end{aligned}$$

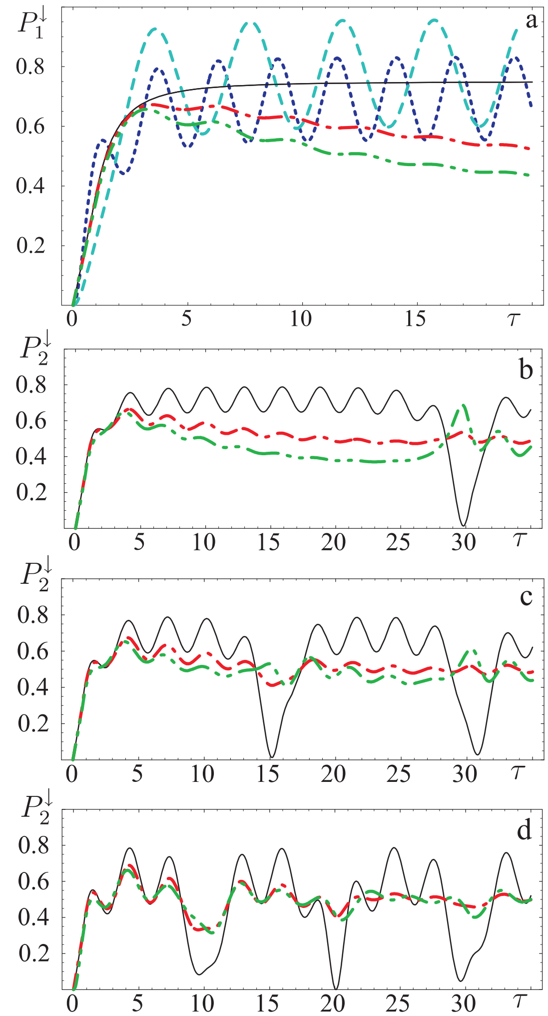

FIG. 1: (Color online) Time dependence of clockwise current direction probabilities. Dot-dashed and double-dot-dashed (red and green) lines show the relaxation and dephasing effects compared to the closed system, solid, dotted and dashed (black, violet and blue) lines. (a) Evolution of $P_1^{\downarrow}$ probability at $\delta = \sqrt{3}/2$ solid, dot-dashed and double-dot-dashed (black, red and green) lines, $\Gamma_r = \Gamma_\varphi = 0.05$ dot-dashed (red) line and $\Gamma_r = \Gamma_\varphi = 0.1$ double-dot-dashed (green) line; $\delta = \sqrt{3}/2 \pm 0.25$ dotted and dashed (violet and blue) lines resp. (b,c,d) Evolution of $P_2^{\downarrow}$ probability at $\delta = \sqrt{3}/2 + 0.1$, $\Gamma_\varphi = 0.1$, $\Gamma_r = 0.05$ dot-dashed (red) line and $\Gamma_r = 0.2$ double-dot-dashed (green) line; (b) $ø = 0.105$, (c) $ø = 0.205$ and (d) $ø = 0.314$.

To show the possibility of controlling the qubit state by external field (7) we plotted function $P_2^{\downarrow}(\tau)$ for δ close to its critical value and for different values of $ø$. Since the above considered case corresponds to $ø = 0$, we show in Fig. 1b (solid (black) line) its behavior for $ø = 0.105$ which is rather close to zero. During a sufficiently long time interval the probability oscillates between 0.6 and 0.8 after which it falls to zero. Closer to zero $ø$ is longer this period becomes and closer to 1/2 $ø$ becomes more this interval is reduced (see the solid (black) lines in

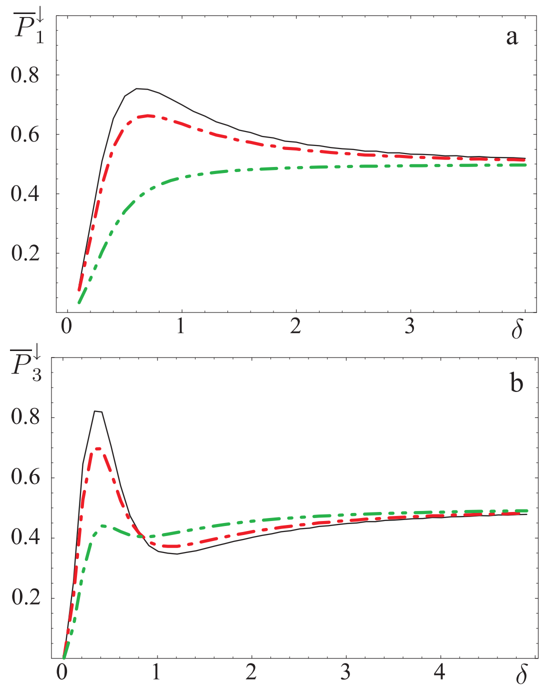

FIG. 2: (Color online) δ-dependence of time-averaged probabilities for the closed system solid (black) lines and with $\Gamma_r = \Gamma_\varphi = 0.01$ dot-dashed (red) lines; (a) $\Gamma_r = \Gamma_\varphi = 0.1$ and (b) $\Gamma_r = \Gamma_\varphi = 0.05$ double-dot-dashed (green) lines.

Figs. 1b, 1c and 1d). We have to notice that in the limit $\o \to 1/2$ ($b \to 0$ in Eq. (7)) function $\epsilon_2(\tau)$ tends to a constant value equal 1/2. This signal reproduces the Rabi oscillations with the frequency $2\sqrt{\delta^2 + 1/4}$. Thus, $\o$ may be considered as a continuously tunable parameter of the external field (7) with the help of which, starting with the usual Rabi oscillations, one may fix the clockwise current direction as long as desirable. We would like to stress that the range of δ, for which the probability exceeds the value 1/2 is rather large, i.e. $0.6 < \delta < 1.1$. This may facilitate its experimental detection.

Analytic expression for $P_2^{\downarrow}(\tau)$ shows us that it is governed by a complicated superposition of two oscillating functions with frequencies θ and $2\o$. Therefore if these frequencies are close enough to each other one may observe a beating phenomenon (Fig. 3). In this case the oscillations with the small amplitude and the frequency close to the Rabi frequency take place at the background of the oscillations with the amplitude close to 1 and the very small frequency defined by the difference between $2\o$ and θ.

Another aspect we would like to emphasize is that this type of the external field is not unique. As we shall now show, there exist other possibilities for the time dependence of the external field exhibiting a similar feature.

Consider an exactly solvable model with a bit more complicated form of function $\epsilon(\tau) = \epsilon_3(\tau)$ [9]

$$\epsilon_3(\tau) = -\frac{1}{2} + \frac{6}{Q_0}\left(\tau^4 + 6\tau^2 - 3\right), \tag{8}$$

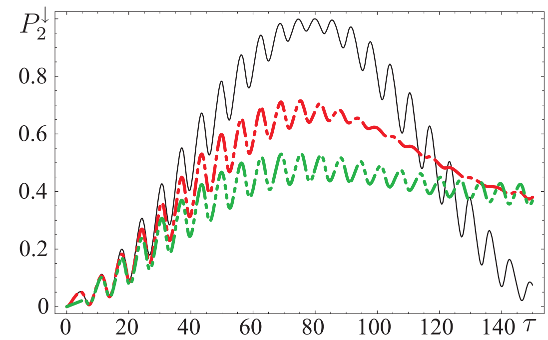

FIG. 3: (Color online) Time dependence of clockwise current direction probabilities at $\o = 0.49$ and $\delta = 0.1$ solid (black line, $\Gamma_r = \Gamma_\varphi = 0.01$ dot-dashed (red) line and $\Gamma_r = \Gamma_\varphi = 0.02$ double-dot-dashed (green) line.

where $Q_0 = \tau^6 + 3\tau^4 + 27\tau^2 + 9$. The clockwise current direction probability $P^{\downarrow}(\tau) = P_3^{\downarrow}(\tau)$ in this case reads

$$\begin{aligned} P_3^{\downarrow}(\tau) &= \frac{4(\theta^2-1)\tau^2}{\theta^8 Q_0} \\ &\times \left[144(1+\tau^2) + \theta^4(\tau^2+9)^2 - 24\theta^2(5\tau^2+9)\right] \\ &+ \frac{(\theta^2-1)Q_1}{\theta^{10}Q_0}\left[Q_2\sin^2(\tfrac{1}{2}\tau\theta) + Q_3\sin(\tau\theta)\right] \end{aligned} \tag{9}$$

where

$$\begin{aligned} Q_1 &= \left[(\theta+1)^2 - 5\right]\left[(\theta-1)^2 - 5\right], \\ Q_2 &= \theta^4 Q_0 + 144(1+\tau^2) - 12\theta^2(5\tau^4 + 6\tau^2 + 9), \\ Q_3 &= 6\theta\tau\left[\theta^2(\tau^4 + 2\tau^2 + 9) - 12(1+\tau^2)\right]. \end{aligned}$$

We notice that this is just the second term in the right hand side of Eq. (9) which is responsible for time-oscillations. Therefore if $Q_1 = 0$ the oscillations in the time-dependence of the probability disappear and once again it acquires a monotonous character. But this time since $Q_1(\theta)$ is a bi-quadratic function, in contrast to the previous case, the clockwise current direction probability turns from an oscillating to monotonous character at two possible values of parameter θ, $\theta = \sqrt{5} \pm 1$. In these cases the behavior of probabilities $P_3^{\downarrow}(\tau)$ and $P_3^{\uparrow}(\tau)$ is illustrated in Figs. 4a and 4b (respective solid (black lines). We thus observe for $P_3^{\downarrow}(\tau)$ an effect similar to that described above for $P_1^{\downarrow}(\tau)$ and, in a sense, the opposite behavior of $P_3^{\uparrow}(\tau)$. The existence of the critical value is reflected also by the averaged probability $\overline{P}_3^{\downarrow}$ which is plotted in Fig. 2b (solid (black) line). It also has a simple analytic expression

$$\overline{P_3^{\downarrow}} = 2\delta^2 \frac{13 - 8\delta^2 + 16\delta^4}{(1+4\delta^2)^2} \tag{10}$$

with a maximum $\overline{P_3^{\downarrow}} \approx 0.91$ at $\delta \approx 0.34$.

The next question we study is how the effect we are observing for an idealized closed system is influenced by

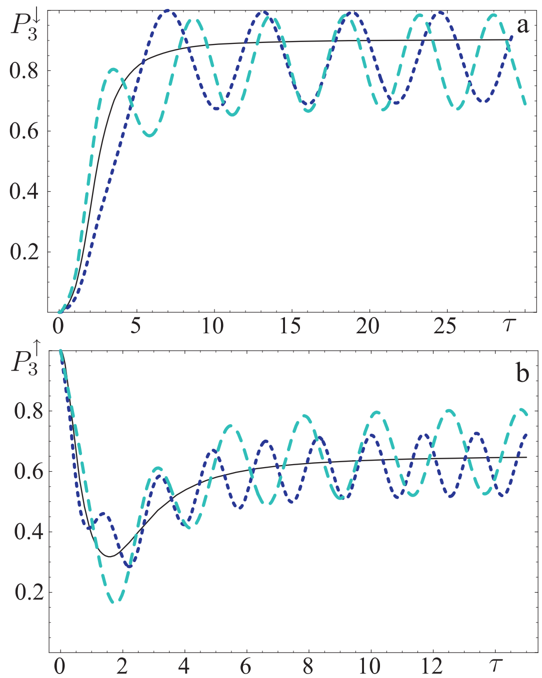

FIG. 4: (Color online) Probabilities (a) $P_3^{\downarrow}(\tau)$ at $\theta = \sqrt{5} - 1$ solid (black) line, $\theta = \sqrt{5} - 1.1$ dotted (violet) line and $\theta = \sqrt{5} - 0.9$ dashed (blue) line; (b) $P_3^{\uparrow}(\tau)$ at $\theta = \sqrt{5} + 1$ solid (black) line, $\theta = \sqrt{5}+1.5$ dotted (violet) line and $\theta = \sqrt{5}+0.5$ dashed (blue) line.

a dissipation featuring open quantum systems [12]. To make rough estimations we are using a phenomenological approach in the density matrix formalism (see e.g. [13]). In this approach a weak coupling of a system to the environment is described by two parameters, dephasing Γ_φ and relaxation Γ_r rates. For the density matrix of the form (cf. [14])

$$\rho = \frac{1}{2}\begin{bmatrix} 1+Z & X-iY \\ X+iY & 1-Z \end{bmatrix}$$

under the initial condition that $2\rho(0)$ is the identity matrix the elements of the density matrix satisfy the Bloch equations [13]

$$\begin{aligned} \dot{X} &= -2\varepsilon(t)Y - \Gamma_\varphi X, \\ \dot{Y} &= -2\delta Z + 2\varepsilon(t)X - \Gamma_\varphi Y, \\ \dot{Z} &= 2\delta Y - \Gamma_r\,(Z - Z(0))\,. \end{aligned}$$

Probabilities $P^{\downarrow,\uparrow}$ are defined only by the diagonal entries of the density matrix, $P^{\downarrow,\uparrow} = (1 \mp Z(t))/2$.

The relaxation and dephasing effects are shown in Figs. 1–3 by dot-dashed (red) and double-dot-dashed (green) lines. As expected, they disturb the system. The influence of dephasing is more crucial and it should not exceed 5 percent of δ value for bias (7). We observe an interesting phenomenon concerning the relaxation. When the probability falls to zero the relaxation smoothes this behavior and can even revert it (see green line in Fig. 1b). Thus, here the relaxation may be considered as helping to keep the state of the qubit unchanged.

Concluding remarks. We have shown that for a qubit placed in a special time-dependent external field there exists a critical value of the tunnel frequency. When the tunnel frequency is close enough to its critical value one may tune the external field frequency in a way to keep the probability value of the clockwise current circulating in the Josephson-junction circuit exceeding 1/2 as long as desirable. Sufficiently small dissipation does not disturb much the phenomenon.

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